Matrix Algebra

Exercises

Teacher prof. dr. AHC van Kampen (Antoine)
a.h.vankampen@amc.uva.nl

Bioinformatics Laboratory, Academic Medical Center
www.bioinformaticslaboratory.nl

Biosystems Data Analysis group, University of Amsterdam
www.bdagroup.nl

Amsterdam, the Netherlands

Exercise 1

Give a null vector of dimension 3.

Exercise 2

Add the following two vectors

$A^T=(2,3,4,1)$, $B^T=(5,-2,3,7)$

$A^T=(1,1,1,1)$, $B^T=(-1,-1,-1,1)$

Exercise 3

Perform the following scalar multiplication

$2A$, given that $A^T=(25,20,5)$

Exercise 4

Add the two matrices

$$A=\begin{pmatrix}5 & 2 & 3\\ 1 & 2 & 7\end{pmatrix}$$

$$B=\begin{pmatrix}6 & 7 & -2\\ 3 & 5 & 19\end{pmatrix}$$

Exercise 5

Substract matrix B from matrix A

$$A=\begin{pmatrix}5 & 2 & 3\\ 1 & 2 & 7\end{pmatrix}$$

$$B=\begin{pmatrix}6 & 7 & -2\\ 3 & 5 & 19\end{pmatrix}$$

Exercise 6

Perform the follow multiplication of a scalar k and matrix A (kA)

$$A = \begin{pmatrix} 2.1 & 3 & 2 \\ 5 & 1 & 6 \end{pmatrix}$$

$$k = 2$$

Exercise 7

multiply matrix A with B (AB)

$$A = \begin{pmatrix} 5 & 2 & 3 \\ 1 & 2 & 7 \end{pmatrix}$$

$$B = \begin{pmatrix} 3 & -2 \\ 5 & -8 \\ 9 & -10 \end{pmatrix}$$

Exercise 8

Find the linear combinations A-B and $A+B-3C$ given that

$A^T=(2,3,6)$

$B^T=(1,1,2)$

$C^T=(10,1,2)$

Exercise 9

What is a linear combination of matrices. Find $A+2B-0.5C$.

$$A = \begin{pmatrix} 5 & 6 & 2 \\ 3 & 2 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 2.1 & 3 & 2 \\ 5 & 1 & 6 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 2.2 & 2 \\ 3 & 3.5 & 6 \end{pmatrix}$$

Exercise 10

Illustrate that $AB \neq BA$.

$$A = \begin{pmatrix} 6 & 3 \\ 2 & 5 \end{pmatrix}$$

$$B = \begin{pmatrix} -3 & 2 \\ 1 & 5 \end{pmatrix}$$

Exercise 11

Show that $C=AI=IA=A$ for every nxn matrix A (I=identity matrix)

Exercise 12

Find the transpose of.

$$A = \begin{pmatrix} 25 & 20 & 3 & 2 \\ 5 & 10 & 15 & 25 \\ 6 & 16 & 7 & 27 \end{pmatrix}$$

Exercise 13

The sales of tires are given by make (rows) and quarters (columns).

$$A = \begin{pmatrix} 25 & 20 & 3 & 2 \\ 5 & 10 & 15 & 25 \\ 6 & 16 & 7 & 27 \end{pmatrix}$$

where the rows present sale of Tirestone, Michigan and Cooper tires, and the columns represent the quarter number 1,2,3,4.

Find the total yearly revenue if the prices of tires vary by quarters as follows

$$B = \begin{pmatrix} 33.25 & 30.01 & 35.02 & 30.05 \\ 40.19 & 38.02 & 41.03 & 38.23 \\ 25.03 & 22.02 & 27.03 & 22.95 \end{pmatrix}$$

Exercise 14

Find the dot product of the vectors, and what is the angle between these two vectors. What is the projection of *A* onto *B*?

$A^T=(4,1,2,3)$

$B^T=(-3,1,7,2)$

Exercise 15
What is the determinant of the following matrices

$$A = \begin{pmatrix} 1 & 3 \\ 5 & 2 \end{pmatrix}$$

$$B = \begin{pmatrix} 20 & 6 & 1 \\ 7 & 8 & 3 \\ 100 & 12 & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 2 & 6 & 3 \\ 0 & 3 & 7 & 4 \\ 0 & 4 & 9 & 5 \\ 0 & 5 & 2 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 2 & 1 & 4 \\ 3 & 2 & 6 \\ 5 & 4 & 10 \end{pmatrix}$$

Exercise 16
Are the three vectors
$A^T=(25,64,144)$
$B^T=(5,8,12)$
$C^T=(1,1,1)$
linearly independent?

Exercise 17
Are the three vectors
$A^T=(1,2,5)$
$B^T=(2,5,7)$
$C^T=(6,14,24)$
linearly independent?

Exercise 18
What is the rank of the following set of vectors
$A^T=(25,64,144)$
$B^T=(5,8,12)$
$C^T=(1,1,1)$

Exercise 19
What is the rank of the following set of vectors
$A^T=(25,64,89)$
$B^T=(5,8,13)$
$C^T=(1,1,2)$

Exercise 20

Show if $B = \begin{pmatrix} 3 & 2 \\ 5 & 3 \end{pmatrix}$ is the inverse of $A = \begin{pmatrix} -3 & 2 \\ 5 & -3 \end{pmatrix}$

Exercise 21

Find the inverse of *A* by the determinant method:

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

Exercise 22

Find the eigenvalues of the matrix

$$\begin{pmatrix} 3 & -1.5 \\ -0.75 & 0.75 \end{pmatrix}$$

Exercise 23

Find the eigenvectors of the matrix

$$\begin{pmatrix} 3 & -1.5 \\ -0.75 & 0.75 \end{pmatrix}$$

Exercise 24

Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}$$

Exercise 25

Find the eigenvalues and eigenvectors of the matrix.
This exercise is more difficult. Only solve if you have sufficient time.

$$A = \begin{pmatrix} 1.5 & 0 & 1 \\ -0.5 & 0.5 & -0.5 \\ -0.5 & 0 & 0 \end{pmatrix}$$

Note that the roots of

$$-\lambda^3 + 2\lambda^2 - 1.25\lambda + 0.25$$

are

$$-(x-0.5)^2(x-1)$$

Exercise 26

Check whether the following matrices are positive (semi) definite, negative (semi) definite or none of the above.

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$$

$$A = \begin{pmatrix} -2 & 4 \\ 4 & -8 \end{pmatrix}$$

$$A = \begin{pmatrix} -2 & 2 \\ 2 & -4 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & 4 \\ 4 & 3 \end{pmatrix}$$

Mathematical topics

Exercises

Teacher prof. dr. AHC van Kampen (Antoine)
a.h.vankampen@amc.uva.nl

Bioinformatics Laboratory, Academic Medical Center
www.bioinformaticslaboratory.nl

Biosystems Data Analysis group, University of Amsterdam
www.bdagroup.nl

Amsterdam, the Netherlands

Exercise 1

Consider the following discontinuous function

$$f(x_1, x_2) = \begin{cases} x_1^2 + x_2^2 & \text{if } x_1 > 1 \\ \left(x_1^2 + x_2^2\right) - 15 & \text{otherwise} \end{cases}$$

Make a drawing of this multivariate function and its contour plot.

Exercise 2

Find the derivatives of

$$f(x) = \frac{ax+b}{cx+d}$$

$$f(x) = \frac{\sqrt{x}}{x^2 - x + 3}$$

Exercise 3

Solve $f'(x)=0$ for $f(x) = \frac{x^2 - x + 5}{2x+3}$

Exercise 4

Find the points on the graph of $y = x^{3/2} - x^{1/2}$ at which the tangent line is parallel to the line $y+2x = 1$.

Exercise 5

Find the derivative of $y=\sin(5x)$ and $y=\ln(17-x)$

Exercise 6

Find the partial derivatives and gradient of $f(x, y) = x^2 y - 4x^2 y^2$

Exercise 7

Find the directional derivative of $f(x, y) = x^2 + 2y^2$ at point *(x,y)=(2,3)* in the direction of $u = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

Exercise 8

Calculate the Hessian matrix of $F(x) = x^2 + 2y^2 + xy$.

Exercise 9

Determine the first four terms in the Taylor expansion of $f(x)=a^x$ at $x=0$.

Introduction to local optimization

Exercises

Teacher prof. dr. AHC van Kampen (Antoine)
a.h.vankampen@amc.uva.nl

Bioinformatics Laboratory, Academic Medical Center
www.bioinformaticslaboratory.nl

Biosystems Data Analysis group, University of Amsterdam
www.bdagroup.nl

Amsterdam, the Netherlands

Exercise 1

(a) Obtain expressions for the gradient and the Hessian of the function
$f(x) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2$.

(b) Verify that $x^* = (1,1)^T$ satisfies $\nabla f(x^*) = 0$ and $\nabla^2 f(x^*)$ positive definite.

(c) Show that $\nabla^2 f(x^*)$ is singular if and only if x satisfies the condition $x_2 - x_1^2 = 0.005$.

Exercise 2

Find the stationary points of the function

$$f(x, y) = 2x^3 - 3x^2 - 6xy(x - y - 1) .$$

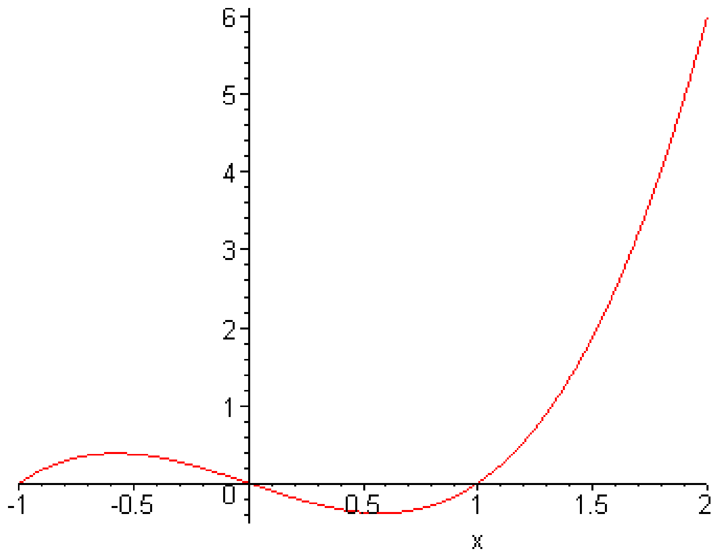
Which of these points or local minima, which are local maxima and which are neither?

Exercise 3

Show that the function $f(x, y) = (y - x^2)^2 + x^5$ has only one stationary point which is neither a local maximum nor a local minimum.

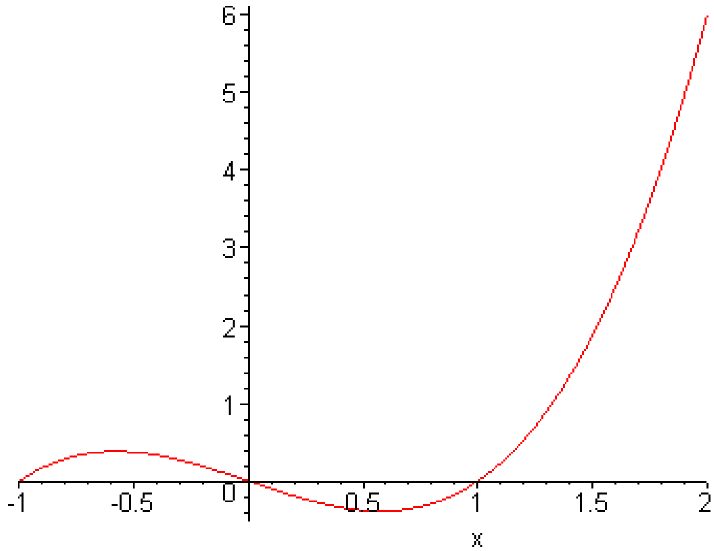
Exercise 4

Determine the equation of the tangent line that gives the slope at point x for the function $y=x^3-x$.



Exercise 5

Use the Newton method for root finding to determine the most right root of $y=x^3-x$. Start at $x=2$. For the first three iterations draw the tangent line in the graph and give the equations of the tangent lines (as determined in the previous exercise) for the first four iterations.



Exercise 6

Consider the following function to be minimized

$$f(x) = -4 + x + \frac{1}{x}.$$

Determine the minimum of this function by differentiation.

Find the value of x which minimizes this function to an accuracy (final interval length) of 0.1 by using

(a) Golden section search
(b) Fibonacci search.

Local Optimization

Exercises

Teacher prof. dr. AHC van Kampen (Antoine)
a.h.vankampen@amc.uva.nl

Bioinformatics Laboratory, Academic Medical Center
www.bioinformaticslaboratory.nl

Biosystems Data Analysis group, University of Amsterdam
www.bdagroup.nl

Amsterdam, the Netherlands

Exercise 1

Consider the function $f(x,y)=5x^2+y^2$.

a) Reformulate this function as $f(x,y)=\frac{1}{2}x^T Ax+b^T x+c$. That is, determine *A, b* and *c*.

b) Since the function *f(x,y)* is a quadratic we can exactly determine the step size α in a steepest descent approach. Perform 2 iterations of the steepest descent procedure. Start from $x_0=(2,2)$.

Now suppose that we don't have an expression to calculate the step size α (which is generally the case for non-linear functions). To minimize the function we must follow a steepest descent procedure with a line search at each iteration to determine the optimal step size. For this simple function *f(x,y)* we can easily determine the step size as this corresponds with the minimum of the 1-dimensional function defined along the line in the direction of the negative gradient.

c) Calculate the gradient of *f(x,y)*. Evaluate the gradient at $x_0=(2,2)$.

d) Determine the equation of the line in the direction of the gradient (thus in the xy-plane)

e) The equation of the line is in terms of *x* and *y*. However, as we are interested in the form of the function along this line. Give the expression of this function and determine the minimum of this function.

Exercise 2

Consider again the function $f(x,y)=5x^2+y^2$. Calculate two steps of conjugated gradients.

Exercise 2

Consider again the function $f(x,y)=5x^2+y^2$. Find the minimum of this function by using the Newton method. Start at $x_0=(2,2)$.

Exercise 4

Consider the function $f(x_1, x_2) = \frac{1}{2}\left(x_1^2 + x_2^2\right)e^{x_1^2 - x_2^2}$.

a) Show that x*=(0,0)^T is a local minimizer of this function
b) Show that the Hessian at point x_0=(1,1)^T is not positive definite.
c) Take one step with the Newton method
d) Show that the Hessian $\nabla^2 f(1,1) + 3I$ is positive definite
e) Take one step with the Newton method and the Hessian $\nabla^2 f(1,1) + 3I$. That is, calculate $x_{k+1} = x_k - \left(G_k^{-1}(1,1) + 3I\right) g_k(1,1)$.