

Very short introduction in linear algebra

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Scalars, Vectors and Matrices

Scalars. A scalar is a single quantity or measurement

Vectors. Ordered collection of scalars

$$a = \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} \longrightarrow \text{Vector transpose: } a = (3, 5, 1)^T$$

$n \times 1 \Leftrightarrow 3 \times 1$

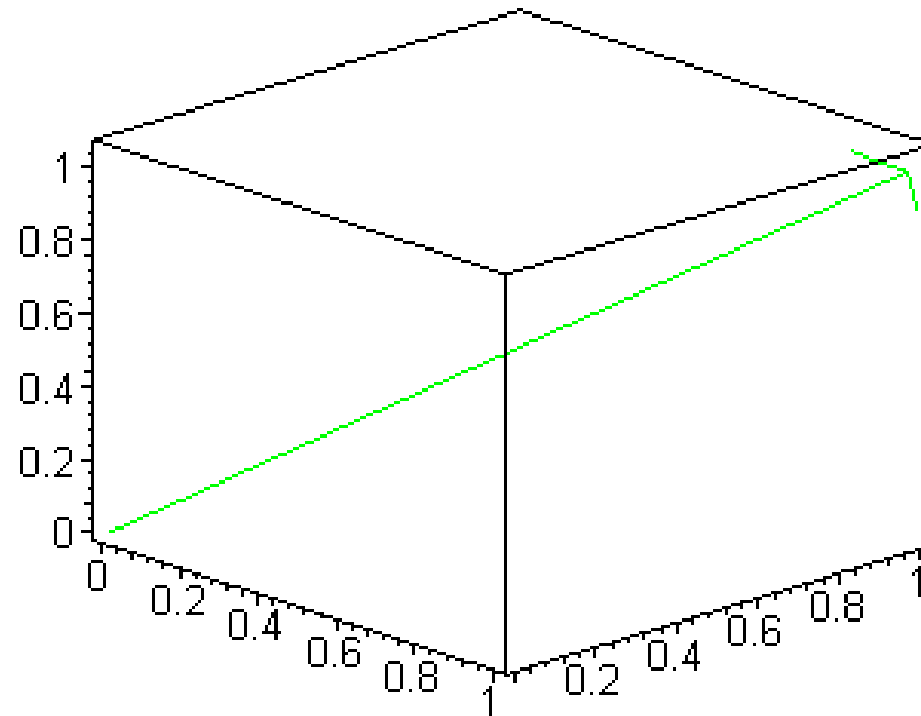
Matrix. Two-dimensional collection of scalars or a collection of column-vectors or row-vectors

$$A = \begin{pmatrix} 1 & 4 & 9 \\ 2 & 4 & 8 \\ 5 & 7 & 1 \end{pmatrix} \quad A^T = \begin{pmatrix} 1 & 2 & 5 \\ 4 & 4 & 7 \\ 9 & 8 & 1 \end{pmatrix} \quad \text{Matrix transpose: } (A^T)_{ij} = A_{ji}$$

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 4 & 5 & 4 \end{pmatrix} \quad A^T = \begin{pmatrix} 1 & 4 \\ 3 & 5 \\ 2 & 4 \end{pmatrix} \quad \text{Matrix trace: } tr(A) = \sum_{i=1}^n a_{ii}$$

Example

$$a=(1,1,1)$$



Vector and matrix dimensions

$$a = \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}$$

$$n \times 1 \Leftrightarrow 3 \times 1$$

$$n \times m \Leftrightarrow 3 \times 3$$

$$A = \begin{pmatrix} 1 & 4 & 9 \\ 2 & 4 & 8 \\ 5 & 7 & 1 \end{pmatrix}$$

$$A^T = \begin{pmatrix} 1 & 2 & 5 \\ 4 & 4 & 7 \\ 9 & 8 & 1 \end{pmatrix}$$

$$n \times m \Leftrightarrow 3 \times 3$$

$$n \times m \Leftrightarrow 2 \times 3$$

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 4 & 5 & 4 \end{pmatrix}$$

$$A^T = \begin{pmatrix} 1 & 4 \\ 3 & 5 \\ 3 & 4 \end{pmatrix}$$

$$n \times m \Leftrightarrow 3 \times 2$$

Multiplication with scalar

$$k \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} kx \\ ky \\ kz \end{pmatrix}$$

$$k \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{pmatrix} = \begin{pmatrix} ka_1 & ka_2 \\ kb_1 & kb_2 \\ kc_1 & kc_2 \end{pmatrix}$$

Vector and matrix addition and subtraction

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix}$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} + \begin{pmatrix} E & F \\ G & H \end{pmatrix} = \begin{pmatrix} A + E & B + F \\ C + G & D + H \end{pmatrix}$$

It is easy to see the vector/matrix addition satisfies the same properties as scalar addition:

Associativity $A + (B + C) = (A + B) + C$

Commutativity $A + B = B + A$

Zero vector and matrix

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

Dot product

The dot product is the scalar product of 2 n -vectors

$$c = a_1b_1 + a_2b_2 + \dots + a_nb_n = \sum_{i=1}^n a_ib_i$$

$$= (a_1, a_2, \dots, a_n) \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = a^T b$$

For example, if

$$a^T = (1, -1, 2) \text{ and } b^T = (1, -1, -1)$$

then

$$a^T b = 1.1 + -1.-1 + 2.-1 = 0$$

Dot product

- If $a^T b = 0$, the vectors a and b are said to be orthogonal.

For example $a = (1, 0)^T$, $b = (0, 1)^T$

- Note also that if $a^T c = b^T c$ it is incorrect (in general) to conclude that $a = b$.

For example consider $a^T = (4, 1)$, $b^T = (2, 3)$ and $c^T = (1, 1)$.

- By the distributivity property, however, it is correct to conclude that $(a - b)^T c = 0$, and hence that the vector $a - b$ is orthogonal to c .

- Only the zero vector is orthogonal to all vectors since $a^T 0$ is always zero.

Matrix multiplication

The product C of two matrices A and B is a matrix defined as follows:

- The (i,j) -the element of C is the scalar product of the i -th row of A and the j -th column of B
- This definition implies that the column dimension of A must be equal to the row dimension of B
- The row dimension of C is the row dimension of A and the column dimension of C is the column dimension of B
- Thus $A(n \times m)B(m \times k) = C(n \times k)$

$$C = AB = \begin{pmatrix} a_1^T \\ a_2^T \\ \vdots \end{pmatrix} (b_1 \quad b_2 \quad \dots) = \begin{pmatrix} a_1^T b_1 & a_1^T b_2 & \dots \\ a_2^T b_1 & a_2^T b_2 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

Matrix multiplication

Example $C = AB = \begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 0 & 2 \end{pmatrix}$

$$C = AB = \begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 0 & 2 \end{pmatrix}$$

$$C = AB = \begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix}$$

$$C = AB = \begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 0 & 2 \end{pmatrix}$$

$$C = AB = \begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} 4 & 3 \\ 8 & -4 \end{pmatrix}$$

Matrix multiplication

- Matrix multiplication satisfies the following properties
 - Associativity** $(AB)C = A(BC)$
 - Distributivity** over matrix addition $A(B+C) = AB + AC$
- However, matrix multiplication is not commutative in general because of the non-interchangeable roles of rows and columns
- Even if the matrices are square (the only case in which the dimensions of AB would match BA), in general $AB \neq BA$
- An additional property of matrix multiplication is that $(AB)^T = B^T A^T$

Identity matrix ($n \times n$)

$$I_n = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

The role of the identity matrix is identical to that of the scalar '1'.

Identity matrix

For A is $m \times n$, I_n satisfies $AI_n = A$

$$C = AI = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$